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RESEARCH ARTICLE

Correlation and the pricing of risks

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Abstract Given a pricing kernel we investigate the class of risks that are not priced by this kernel. Risks are random payoffs written on underlying uncertainties that may themselves either be random variables, processes, events or information filtrations. A risk is said to be not priced by a kernel if all derivatives on this risk always earn a zero excess return, or equivalently the derivatives may be priced without a change of measure. We say that such risks are not kernel priced. It is shown that reliance on direct correlation between the risk and the pricing kernel as an indicator for the kernel pricing of a risk can be misleading. Examples are given of risks that are uncorrelated with the pricing kernel but are kernel priced. These examples lead to new definitions for risks that are not kernel priced in correlation terms. Additionally we show that the pricing kernel itself viewed as a random variable is strongly negatively kernel priced implying in particular that all monotone increasing functions of the kernel receive

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a negative risk premium. Moreover the equivalence class of the kernel under increasing monotone transformations is unique in possessing this property.

Keywords Kernel pricing · Change of measure · Catastrophic risk pricing · Self sufficient filtrations

JEL Classification Numbers G10 · G12 · G13

1 Introduction

An important question in finance is the identification of risks that are priced in the economy. The traditional approach to this issue has been one of ascertaining whether the covariance of returns on financial assets with the risk in question is priced in a classical analysis of cross-sectional excess returns (Fama 1970). A particular consequence of this method is that risks that are uncorrelated with all financial asset returns, are by virtue of then being untraded, also not priced. However, one recognizes that in complete markets there are no nontrivial risks that are uncorrelated with all financial returns. Alternatively, one has the relatively more recent approach developed in the literature on derivatives where an event is priced if the probability used for pricing differs from the statistical or true probability (Harrison and Kreps 1979; Harrison and Pliska 1981). In the latter case, the Arrow Debreu security associated with this event has a different expectation under the two measures and their ratio reflects the excess return. The two approaches are consistent in recognizing that a risk is priced if it impacts financial excess returns. In particular, we recognize that the latter approach when viewed from the perspective of the former equates the absence of a change of probability, to all functions of the risk in question being uncorrelated with the relevant financial asset return (e.g. the market portfolio in the case of the capital asset pricing model) and hence as earning a zero excess return.

For this paper we define risk as a traded payoff function that is measurable with respect to some information filtration and it is thus a random variable. This payoff random variable, however, has a probability law that is deduced from the underlying risks of the payoff. This underlying risk may be as simple as, for example, the level of an equity index at a future point of time. Alternatively it may be a function of the path as is typical for structured products like cliquets or options on the realized variance. The risk in the latter case depends on the law of the process, in contrast with the former case where we focus only on the underlying law of the random variable defined by the level of the index at the specified time. Apart from levels and paths contracts are also written on specific events like corporate defaults or catastrophic losses associated with events like hurricanes or earthquakes. More generally the underlying risk could be that of a subfiltration where we take a $\mathcal{G}$ measurable random variable for a payoff for a general subfiltration $\mathcal{G}$.

For the pricing of risks or underlying risks we shall henceforth refer to a risk as being kernel priced or not kernel priced if the probability law of this risk in question is changed or not changed by a particular candidate pricing kernel or equivalently some contingent claim written on the risk in question earns a nonzero excess return at some time. In this case we say that the underlying risk is priced.

Increasingly one also sees the pricing kernel explicitly defined in a continuous time model, where the risks are described by the natural filtration associated with the time paths of processes. By way of example we cite Bakshi and Chen (1997), Duffie et al. (2000), Chernov and Ghysels (2000), Collin–Dufresne and Goldstein (2002), Carr et al. (2002) and Kimmel (2004). We wish to analyze the precise relation between the absence of a change of probability and concepts of no correlation in such continuous time models with respect to risks defined by various subfiltrations of the economy filtration. For instance one might ask if firm-specific dividend yields in the Bakshi and Chen (1997) model are not kernel priced. Alternatively, and somewhat more interesting is whether the risk of the time to first default in a basket of names is kernel priced by a particular estimated kernel on the larger filtration of the economy. More importantly, it is essential to be careful about incorrectly infering the absence of a change of probability on the mere confirmation of the absence of correlation between a risk and the pricing kernel.

In fact our investigation was initially motivated by comments in the literature on catastrophic risks (see Cummins et al. 1999; Froot 1999; Doherty 1997) where it is argued that reinsurance contracts on losses related to natural disasters should be priced at the actuarial loss value as the risks on such exposures are not correlated with a wide class of financial assets. These reinsurance contracts are derivative contracts and their actuarial valuation implies that they are to be priced without a change of probability or in our terminology they are not kernel priced by any reasonable kernel. It therefore seemed to us that a critical evaluation of the relationship between the absence of correlation and the absence of a change of probability was called for. Our investigation into these questions, for the purpose of rigor and clarity, took an abstract mathematical turn. However, we provide in Sect. 2 below a commentary on the implications of these results to financial questions related to the pricing of contingent claims.

We recognize that a change of probability is intimately linked with the presence of excess returns, positive or negative. We shall show however that, correlation between processes and the pricing kernel, is an important but incomplete guide to issues of kernel pricing. In fact, we provide examples of kernel pricing in situations of zero correlation. A more detailed investigation of the examples reveals that the absence of kernel pricing is related to the absence of correlation provided the latter tests are conducted with respect to a particular set of terminal random variables. These variables define an extended filtration. This filtration includes the time paths of variables useful in predicting the evolution of the risk in question. We call such filtrations *self sufficient* within the economy filtration and define this concept. A filtration may lose the self sufficiency property under a measure change and in this case the absence of kernel pricing is also lost. Hence we study the conditions under which we have the invariance of self sufficiency under changes of probability. Essentially we demonstrate that self sufficiency will be preserved if the pricing kernel does not have any hidden exposure to risks in the self sufficient filtration in a sense made precise later in the paper.

Additionally we show that the pricing kernel, now itself seen as a random variable is "strongly negatively priced" in that the law of the kernel as a random variable, after the measure change first order stochastically dominates its law before the change. As a consequence all monotone increasing functions of the kernel as a random variable or equivalently all random variables aligned with the kernel receive a negative risk premium. Moreover, the kernel (viewed as an equivalence class of random variables under increasing monotone functional transformations) uniquely possesses this property and hence it constitutes the classic insurance asset for the economy for which it is the kernel. The contribution of this paper is threefold: (1) we provide a precise definition of the concept of risks that are not kernel priced, (2) we demonstrate that the pricing kernel is the unique equivalence class of random variables with the property that all random variables aligned with it receive a negative risk premium, and (3) we introduce the concept of self sufficient filtration and investigate its role in kernel pricing.

Two sets of correlation tests result. The first addresses no kernel pricing assuming that the chosen self sufficient filtration containing the risk filtration remains self sufficient under the measure change and hence we may evaluate conditional expectations restricted to this chosen self sufficient filtration. These tests deliver the property of almost no kernel pricing abbreviated as ANKP. The second collection of tests validate the invariance of self sufficiency under the measure change as required for the full no kernel pricing NKP property. As self sufficient filtrations containing a particular risk filtration are not unique henceforth when we make a reference to 'the self sufficient filtration' we mean a fixed particular choice from the collection of self sufficient filtrations.

For ANKP one has to ensure that the projection of the pricing kernel onto the self sufficient filtration containing the risk in question is orthogonal to all the martingales in this self sufficient filtration that finish in a random variable measurable with respect to the risk filtration. This requires that the conditional expectation process of a terminal random variable measurable with respect to the risk filtration, which in general is a martingale adapted to the self sufficient filtration, is orthogonal to the projection of the pricing kernel onto this filtration. This is a fairly large class of orthogonality tests that is well defined in one variable, the projection of the pricing kernel, but is large in the class of terminal random variables to be considered.

For NKP one has to further test that the stochastic logarithm of the kernel less the stochastic logarithm of its projection onto the self sufficient filtration is orthogonal to all the martingales in the self sufficient filtration. This is also a second large class of tests. The deviation between the stochastic logarithm of the kernel and its projection is well defined, but the class of self sufficient martingales is now generated by all the terminal random variables of this filtration which is larger than the risk filtration. We expect that in most cases of practical interest self sufficiency under the physical measure will carry over to self sufficiency under the measure change. The pricing kernel has to be very specially designed to violate self sufficiency under the measure change by constructing an exposure of the kernel to the martingales in the filtration self sufficient under the physical measure that has a null projection onto this filtration and therefore is unobservable to the conditional expectation operator and hence we call such exposures hidden exposures. Barring such hidden exposures in the kernel ANKP is in fact NKP.

The development of econometric procedures for the evaluation of the self sufficiency of a filtration with respect to a larger filtration is an important question for future research into issues of risk pricing proposed by our study. In addition we need to address issues of basis generation of terminal random variables in a filtration to exhaust its martingales and make operative the orthogonality tests proposed here in a relatively abstract framework.

The remainder of the paper is organized as follows. Section 2 presents a financial commentary that draws on the results of the rest of the paper and illustrates their use in addressing issues of interest to the wider financial community. Section 3 briefly addresses the case of a continuous state one period model. The continuous time context with risks defined by the filtration associated with a stochastic process is studied in Sect. 4. We present here our result on the strong negative pricing of the change of measure density, a number of examples connecting correlation and pricing, followed by our result relating the absence of kernel pricing to no correlation. In Sect. 5 we study the important question of the invariance of self sufficiency to a change of measure. Section 6 considers event risk in a continuous time context. Section 7 concludes and presents a few remaining questions, which seem to be hard to solve and should be of interest to theorists in financial economics and probability.

2 Financial commentary

We employ, in this section, the results of this paper to comment on a number of issues related to the pricing of financial risks, a prototype being catastrophic risk. We also comment on the role of self sufficient filtrations in asset pricing in the context of the Heston (1993) model. This latter discussion also illustrates how processes adapted to the pricing kernel may fail to undergo a change of law under the measure change defined by the kernel. These issues are taken up in the following three brief subsections.

2.1 Catastrophic risk and measure changes

One may take a stylized view of the effects of catastrophes as driven by a Brownian motion say $(w_1(t), t \geq 0)$ that is orthogonal to a second Brownian motion $(w_2(t), t \geq 0)$ driving the market portfolio or the pricing kernel for that matter. The severity of the impact of catastrophic events depends on things like property values and these may very well be random and related to the market portfolio or the pricing kernel. Hence the integrand of the loss process has a potential dependence on the kernel. The kernel changes very strongly the law of the random variable with which it is associated as shown in Propositions 3 and 4.

Such a view of the situation puts us quite easily in the context of the single period model described by Eq. (2) where we show that there is no reasonable pricing kernel, that is a function of the integrand, that does not change the law of the reinsurance loss process. Hence these considerations suggest that catastrophic loss reinsurance claims should always be priced under a change of measure, notwithstanding claims to the contrary in the existing literature.

2.2 Self sufficiency in the Heston model

The volatility process of the Heston model is an autonomous Markov process under the stated physical law. It can be shown that the volatility filtration is self sufficient in the full filtration under the physical law. We might ask how this self sufficiency may be lost under the risk neutral law. Or in other words, what sorts of market prices of risk would lead to a loss of self sufficiency under the law induced by the resulting kernel?

It may be shown, using Proposition 6, that as long as the price of volatility risk is just a function of the volatility process, self sufficiency is maintained. We note that all the risk pricing formulations studied in the literature make such an assumption and therefore self sufficiency has been preserved. On the other hand, just adapting the market price of volatility risk to the process for the stock price may not in and of itself involve a loss of self sufficiency. For this the kernel has to have a market price of risk which is orthogonal to the density when projected onto the volatility filtration. Alternatively the market price of volatility risk must have a zero conditional expectation when projected onto the volatility process. Hence there is an exposure that is not observed by the volatility process and that is a specially designed and hidden exposure. We have at this writing not constructed such a hidden exposure in this filtration but expect that it is possible.

2.3 Dependence and no change of measure

The technique of hidden exposures is also the vehicle for ways of building exposures to martingales correlated with the pricing kernel whereby the correlation disappears upon projection onto the self sufficient filtration containing the risk filtration. Since the exposure vanishes upon projection, it guarantees that the risk filtration does not see a change of law even though it was partly adapted to risks correlated with the movements of the kernel.

3 Single period risk

Consider a one period two date model with uncertainty resolved at time 1 and prices determined at time 0. For simplicity, we suppose that interest rates are zero. The uncertainty in the economy is given by the probability space $(\sum, \mathcal{F}, P)$, where P is the true probability on the event space $(\sum, \mathcal{F})$. Let X denote a random variable defined on this space. This variable could represent a future asset price or the level of some macroeconomic variable.

We wish to consider in addition, a pricing kernel or a candidate for a change of probability. For the current context of a single period zero interest rate economy, it suffices to define it as a positive random variable Y with unit expectation under P. The random variable Y is also commonly referred to in the literature as a state price density (See for example Campbell et al. (1997) and the references therein). Hence the pricing measure or risk neutral probability of a set $A \in \mathcal{F}$ is given by the probability Q,

$$Q(A) = E^P \left[Y \mathbf{1}_A \right],$$

where E^P denotes expectation under P and $\mathbf{1}_A$ is the indicator function of the set A.

We define the underlying risk of a random variable X as *not kernel priced* (NKP) by the kernel Y if the distributions of X under P and Q are the same. Equivalently, for any contingent claim paying $f(X)$ at time 1, with finite expectation under P, the risk neutral expectation of $f(X)$ equals its statistical expectation or

$$E^Q[f(X)] = E^P[f(X)]$$

and hence there is no risk premium on all such claims. Note finally that one has

$$\text{NKP} \Longleftrightarrow E^P\,[Y|X] = 1. \tag{1}$$

On the other hand, if the distribution of X under Q differs from that under P then for some positive function f , $0 \leq f \leq 1$ of X, we must have $E^Q[f(X)] < E^P[f(X)]$ and in this case there is a positive risk premium for the security paying $f(X)$ and the contingent claim f represents a risk that is being compensated. Likewise, for the positive function $g(X) = 1 - f(X)$ we have the property $E^Q[g(X)] > E^P[g(X)]$ and the contingent claim g represents a hedge or an insurance contract with a negative excess return. Typically, out-of-the-money puts on the market index, say the S&P500 are examples of the latter while at-the-money index calls are examples of the former (see for example Jackwerth 2000).

The first simple observation we make, for emphasis, is that the underlying risk of the random variable X may not have the NKP property but yet may be uncorrelated with the kernel Y. Consider for example the case of X a standard normal variable and take for Y the absolute value of X, scaled to a unit expectation. The expectation of the product of Y and X is then zero and we have a zero covariance. However, X and Y are clearly not independent as the latter is just a functional transformation of the former and we observe that the probability distribution of X under Q has been changed. In fact the probability of $X > 1.96$ under P is 2.5% while under Q this probability is easily computed to be above 7%.

On the other hand for an arbitrary kernel Y, if all contingent claims trade and we have a zero covariance for all positive contingent claims so that

$$E^P[f(X)\,(Y(X)-1)] = 0 \quad \text{for all } f \geq 0$$

then it must be the case that $Y = 1$, $Q = P$ and there is no change of probability and we have the NKP property. Hence we note that a broader test of orthogonality is linked to the absence of a change of probability. The more general results we develop later for filtrations are in this vein.

Yet another example of zero correlation and a change of probability, related to issues of stochastic volatility, arises when we take the risk X to be of the form

$$X = \sqrt{G}Z,$$

where Z is a standard normal variate and G is a positive random variable independent of Z. Here the volatility of the risk conditional on the realization of G is $\sqrt{G}$. To be specific, and to illustrate our point we consider the case when G has the gamma density f_a with parameter a

$$f_a(\gamma) = \frac{\gamma^{a-1}}{\Gamma(a)} \mathrm{e}^{-\gamma}$$

then the conditional characteristic function of X given G is

$$E[\mathrm{e}^{\mathrm{i}uX}|G] = \exp\left(-G\frac{u^2}{2}\right).$$

The unconditional characteristic function is then easily evaluated from the Laplace transform of a gamma variable as

$$E\left[\mathrm{e}^{\mathrm{i}uX}\right] = \left[\frac{1}{1+\frac{u^2}{2}}\right]^a.$$

Now consider the kernel

$$Y = 2^a \mathrm{e}^{-G}$$

then Y and X are uncorrelated but the law of X is changed as the Laplace transform of G under this kernel is

$$\begin{aligned} E^Q\left[e^{-\lambda G}\right] &= E^P[2^a e^{-G} e^{-\lambda G}] \\ &= \left[\frac{2}{2+\lambda}\right]^a. \end{aligned}$$

It follows that the characteristic function of X under Q is then

$$E^Q[e^{iuX}] = \left[\frac{2}{2+\frac{u^2}{2}}\right]^a.$$

Hence with obvious notation

$$X^Q \overset{(d)}{=} \frac{1}{\sqrt{2}} X^P.$$

Somewhat more generally we may consider the context of two independent random variables, Z a standard normal variate and Y a positive random variable with density $q(y)$, and

$$X = \sqrt{Y} Z. \tag{2}$$

In this general context we may show that there is no nontrivial kernel $f(Y)$ for which the law of X is left unchanged. This is done by computing the conditional expectation of $f(Y)$ given X and observing that this could never be 1 for any nonconstant f. To evaluate the conditional expectation we note that for any test function $g(X)$

$$E\left[f(Y)g(X)\right] = \int_{-\infty}^{\infty} dx \int_0^{\infty} dy\, q(y) f(y) \frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}} g(x).$$

But we also have with $h(x)$ the density of X that

$$E\left[f(Y)g(X)\right] = \int_{-\infty}^{\infty} dx\, h(x) E\left[f(Y)|X = x\right] g(x),$$

where

$$h(x) = \int_0^{\infty} dy\, q(y) \frac{1}{\sqrt{2\pi y}} e^{-\frac{x^2}{2y}}.$$

Hence it follows that

$$E\left[f(Y)|X=x\right] = \frac{1}{h(x)} \int_0^\infty \mathrm{d}y\, q(y) f(y) \frac{1}{\sqrt{2\pi y}} \mathrm{e}^{-\frac{x^2}{2y}}.$$

This conditional expectation is unity just if

$$\int_0^\infty \mathrm{d}y\, q(y) \frac{1}{\sqrt{2\pi y}} \mathrm{e}^{-\frac{x^2}{2y}} = \int_0^\infty \mathrm{d}y\, q(y) f(y) \frac{1}{\sqrt{2\pi y}} \mathrm{e}^{-\frac{x^2}{2y}} \tag{3}$$

for all x. We observe that both sides of Eq. (3) are Laplace transforms in $\frac{x^2}{2}$ with respect to the variable $1/y$. By the uniqueness of Laplace transforms we deduce that $f=1, q(y)\mathrm{d}y$ a.e.

A somewhat different construction of NKP and zero correlation is to take for X an exponential random variable with respect to which the Laguerre polynomials are an orthogonal family and in particular we may use the second Laguerre polynomial $L_2(X) = (X-2)^2 - 2$ to define the change of probability by

$$Y = 1 + \frac{1}{2} L_2(X)$$

that satisfies zero correlation with X but changes the law of X. One could use such constructions to leave invariant the first n moments under the change of measure induced by Y and yet change the law of X.

It is correct to observe that in our examples with zero correlation between X and Y we have induced a dependence for if they were independent then there would be no change of law for X as

$$E^P[Y|X] = E^P[Y] = 1 \tag{4}$$

and so we must introduce some dependence and we do. However, it is equally important to note that the NKP property does not mean that Y and X are independent as this requires that for all measurable functions $f, g \geq 0$ we have

$$E[g(Y)f(X)] = E[g(Y)]E[f(X)] \tag{5}$$

and the property (5) is a much stronger statement than NKP or the property (4). In other words it is possible for Y and X to be dependent but yet we have NKP. A simple example is given by

$$Y = \mathrm{e}^{XZ - \frac{X^2}{2}}, \tag{6}$$

where Z is a standard normal variate independent of X a nontrivial (i.e. not equal to a constant almost surely) positive random variable. Clearly Y and X are dependent by construction but as $E[Y|X] = 1$ the NKP condition (1) holds.

On the other hand we recognize that if there is a nonzero covariance then the distribution of X under Q is not the same as under P, in fact the mean changes. Furthermore, we note that if we do have a change in probability then there may yet remain contingent claims written on X that have as random variables in their own right the NKP property. This is simply observed by taking again the standard normal variate X under P and changing probability using $Y = c(X^- + \mathbf{1}_{X \geq 0})$ (where c is a normalization constant) which leaves the law of X^+ unchanged. Such a view towards changes in probability has recently been taken in Jarrow and Purnandam (2004).

The general question we wish to investigate is the structure of risks that have the NKP property and those for which the law is changed by a candidate change of measure. A decomposition of the space of risks into those with the NKP property and its complement would be instructive. In particular, is it possible that the set of nontrivial risks with the NKP property is empty or that every nontrivial risk is priced? In the above example of a standard normal distribution for X under P there is always a nontrivial risk with the NKP property. To construct an element of NKP consider the arbitrary change of measure defined by the non constant function $Y(x)$ such that the positive random variable $Y(X)$ has a unit expectation. The function $Y(x)$ must take values both below and above unity and so there exists proper subsets A, B of the real line such that $E[Y(X)|X \in A] < 1$ and $E[Y(X)|X \in B] > 1$. Let $\mathcal{A} = \{A|E[Y(X)|X \in A] < 1\}$ and let $\mathcal{B} = \{B|E[Y(X)|X \in B] > 1\}$. Let A^* be the union of a maximal chain of elements of $\mathcal{A}$. We must then have that $E\left[Y(X)|X \in A^*\right] = 1$ and as a consequence $E\left[Y(X)|X \in A^{*c}\right] = 1$. Further we note that A^* is a proper subset of the real line. Define the random variable $W = \mathbf{1}_{A^*}$. By construction $E[Y(X)|W] = 1$ and W is a nontrivial risk that has NKP.

For a more concrete example suppose that U is uniform on $[0, 1]$ and $Y(u)$ is $2u$. We can choose $A^* = \left\{u|\frac{1}{4} \leq u \leq \frac{3}{4}\right\}$ and evaluate

$$E\left[Y(u)|u \in A^*\right] = 1.$$

On the other hand for a single period two state model there are no nontrivial risks that are not priced by a measure change that has altered the probability of one of these two states. Essentially there is only one random variable here and it is aligned with the measure change density.

Instead of studying these questions at the level of random variables in a single period context, we recognize that many of the risks that have to be priced in practice involve complex claims referring to time paths of financial or macroeconomic variables. We therefore take up these questions in the more relevant context of continuous time models in the next section.

4 Risk in continuous time

For our true probability space we consider $\left(\sum, \mathcal{F}_T, \{\mathcal{F}_t\}_{0\leq t\leq T}, P\right)$ a filtered probability space where $\mathcal{F} = (\{\mathcal{F}_t\}_{0\leq t\leq T})$ is the information filtration of the economy. The space of stochastic processes that we shall consider consists of $\mathcal{F}$ semimartingales and the concepts we introduce will be defined for such processes. However, in the construction of examples and counterexamples we shall restrict attention to the widely used model of an *n-dimensional* Brownian motion $(B(t), t \leq T)$ and its natural filtration. For general results on continuous martingales and Brownian motion we refer to Revuz and Yor (2001). Well known examples of models in finance formulated with this general structure include the Black and Scholes (1973) and Merton (1973) models for equity options, the jump diffusion model of Merton (1976), the Heath et al. (1992) model for interest rate risk, the Heston (1993) stochastic volatility model, and more recently the Lévy process models of Eberlein et al. (1998) and Carr et al. (2002).

The results of Harrison and Kreps (1979), and Harrison and Pliska (1981) establish the foundations of arbitrage free pricing showing that such prices in a zero interest rate economy are given by expected cash flows evaluated under a change of measure to Q. This change of measure is always given by a positive $\mathcal{F}_T$ measurable random variable D_T that defines the terminal value of the density of the pricing measure Q with respect to the measure P. The change of measure density process $D = (D_t, 0 \leq t \leq T)$ is given by the $(\mathcal{F}, P)$ martingale

$$D_t = E^P\left[D_T|\mathcal{F}_t\right].$$

This density is explicitly modeled in many of the papers already cited and here we reference by way of examples Duffie et al. (2000), Chernov and Ghysels (2000) and Carr et al. (2002). In this context we wish to describe the processes that are kernel priced and those that are not kernel priced and hence have the NKP property at the process level. Specifically we suppose that the pricing kernel is given and may also have been estimated.

Consider for this purpose a set of underlying risks that are specified by an $\mathcal{F}$ semimartingale vector $X = (X_t, 0 \leq t \leq T)$. In general we are interested in the valuation of claims written on the price paths of a set of risk variables and so we consider in general a $\mathcal{X}_T$ measurable functional where $\mathcal{X}_T = \sigma\{X(t), t \leq T\}$ and denote this path functional by $F = F(X(u), u \leq T)$. We also denote by $\mathcal{X}$ the filtration $\mathcal{X} = \{\mathcal{X}_t, t \leq T\}$ where $\mathcal{X}_t = \sigma\{X(s), s \leq t\}$. The prices of contingent claims are developed in a no arbitrage framework by evaluating discounted expected cash flows under a change of probability. For the purposes of this paper we have assumed a zero interest rate environment and hence we consider just the expected cash flow under a change of probability to Q.

At the initial date, prices $\pi_0(F)$ will be given by

$$\pi_0(F) = E^Q\left[F(X(u), u \leq T)|\mathcal{F}_0\right].$$

At subsequent times $s < T$ a conditional expectation is involved in determining the price. In general we condition on all information at time s (e.g. $\mathcal{F}_s$). We then write

$$\pi_s(F) = E^Q\left[F(X(u), u \leq T)|\mathcal{F}_s\right].$$

Consider now the expectation of F at time s evaluated without a measure change. We write this as

$$\mu_s(F) = E^P\left[F\left(X(u), u \leq T\right)|\mathcal{F}_s\right]$$

However, we now note that with respect to claims that are $\mathcal{X}_T$ measurable there is in general a lot of information in $\mathcal{F}_s$ that has nothing to do with the evolution of the paths of $X(t)$. For a particular example, we know that the level of the stock price at time s has no relevance for the evolution of volatility in the Heston (1993) stochastic volatility model under the specified physical probability P. These considerations lead us to introduce the concept of self sufficient filtrations that we now define.

A subfiltration $\mathcal{J}$ is said to be *self sufficient* in $\mathcal{F}$ if every $(\mathcal{J}, P)$ martingale is an $(\mathcal{F}, P)$ martingale. This condition is equivalent to the statement that for every $J_T \geq 0$, $\mathcal{J}_T$ measurable random variable and $s < T$ we have that

$$E^P\left[J_T|\mathcal{F}_s\right] = E^P\left[J_T|\mathcal{J}_s\right], \tag{7}$$

or equivalently that $\mathcal{J}_T$ and $\mathcal{F}_s$ are conditionally independent given $\mathcal{J}_s$ (Brémaud and Yor 1978).

Hence for self sufficient filtrations, all information necessary for predicting a $\mathcal{J}_T$ measurable random variable is already in $\mathcal{J}_s$. The concept of self sufficiency was studied in Brémaud and Yor (1978) where it was referred to as hypothesis H. It was later referred to as $\mathcal{J}$ being immersed in $\mathcal{F}$ in Tsirelson (1998) and Emery (2002). For some examples of a lack of self sufficiency involving Brownian motion see Jeulin (1996). We note here that if $\{\mathcal{B}_t\}$ denotes the one-dimensional Brownian filtartion generated by $(B_t, t \geq 0)$, then this filtration loses self sufficiency with respect to any non-trivially enlarged filtration, that is: $\{\mathcal{B}_t \vee \sigma(X)\}_{t\geq 0}$, for X a non constant $\mathcal{B}_\infty$-*measurable* variable. Indeed, if (B_t) remained a martingale with respect to $\{\mathcal{B}_t \vee \sigma(X)\}$, then it would be a Brownian motion in that filtration and therefore would be independent of the information at the origin of time, that is $\sigma(X)$, which is absurd since X is a nontrivial functional of (B_t). For a further reference on initial enlargements we mention Mansuy and Yor (2006).

For evaluating conditional expectations we may without loss of generality focus attention on self sufficient filtrations. We denote by $\mathcal{Z}$ any filtration containing $\mathcal{X}$ with the property that $\mathcal{Z}$ is self sufficient in $\mathcal{F}$. We always assume that

$\mathcal{Z}_0$ is trivial. The expectation of the claim F at time s is then

$$\begin{aligned}\mu_s(F) &= E^P\left[F(X(u), u \leq T)|\mathcal{F}_s\right] \\ &= E^P\left[F\left(X(u), u \leq T\right)|\mathcal{Z}_s\right].\end{aligned}$$

We say that X has the NKP property with respect to D if for all functionals of the paths of X , $F = F(X(u)_{0\leq u\leq T})$ it is the case that for all $s < T$

$$\pi_s(F) = \mu_s(F) \tag{8}$$

$$E^Q\left[F(X(u), 0 \leq u \leq T)|\mathcal{F}_s\right] = E^P\left[F(X(u), 0 \leq u \leq T)|\mathcal{Z}_s\right] \tag{9}$$

We note that expected values are computed in accordance with the right hand side of (9). The price is given by the left hand side of (9). When NKP holds, all contingent claims written on the paths of X have a zero excess return continuously through time when the pricing kernel is D_T. In this case all the X contingent claims are not priced in the sense of earning zero excess returns continuously through time for investments over arbitrary horizons on any of the derivative securities. The equivalence of (9) with (8) is based on exploting the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ as we may then replace conditioning by $\mathcal{Z}_s$ on the right hand side of Eq. (9) by conditioning on $\mathcal{F}_s$. We make a detailed study of this NKP property (8) in Sects. 4.2 and 5 of this paper.

For ease of notation we shall delete the superscript P on the expectation operator and we understand that all expectations, when no measure is indicated, are under the measure P.

We now provide an important necessary condition for X to possess the NKP property under the additional hypothesis that $\mathcal{X}$ is itself self sufficient in $\mathcal{F}$.

Proposition 1 *If X has the NKP property and $\mathcal{X}$ is itself self sufficient in $\mathcal{F}$ under P then*

$$E\left[D_T|\mathcal{X}_T\right] = 1.$$

where the expectation here is under the measure P.

Proof Suppose X has the NKP property with respect to D_T. We apply the property (9) to the claim given by $F = E[D_T|\mathcal{X}_T]$ to deduce on noting $\mathcal{Z}_s = \mathcal{X}_s$ that

$$E\left[D_T E[D_T|\mathcal{X}_T]|\mathcal{F}_s\right] = E\left[E[D_T|\mathcal{X}_T]|\mathcal{X}_s\right]. \tag{10}$$

It follows in particular that the left hand side is now $\mathcal{X}_s$ measurable and so

$$\begin{aligned}E\left[D_T E[D_T|\mathcal{X}_T]|\mathcal{F}_s\right] &= E\left[D_T E[D_T|\mathcal{X}_T]|\mathcal{X}_s\right] \\ &= E\left[\left(E\left[D_T|\mathcal{X}_T\right]\right)^2|\mathcal{X}_s\right].\end{aligned} \tag{11}$$

Combining Eqs. (10) and (11) we deduce that

$$E\left[\left(E\left[D_T|\mathcal{X}_T\right]\right)^2|\mathcal{X}_s\right] = E\left[E[D_T|\mathcal{X}_T]|\mathcal{X}_s\right].$$

Taking expectations on both sides, we get:

$$\begin{aligned} E\left[\left(E\left[D_T|\mathcal{X}_T\right]\right)^2\right] &= E\left[E[D_T|\mathcal{X}_T]\right] \\ &= E[D_T] = 1 \\ &= \left(E\left[E[D_T|\mathcal{X}_T]\right]\right)^2. \end{aligned}$$

Hence $E[D_T|\mathcal{X}_T]$ must be a constant equal to its expectation that is unity. □

This necessary condition for the NKP property when $\mathcal{X}$ is self sufficient in $\mathcal{F}$ does not on its own imply the NKP property. From this necessary condition we may deduce that

$$\begin{aligned} E^Q\left[F\left(X(u), u \leq T\right)|\mathcal{X}_s\right] &= E\left[D_T F\left(X(u), u \leq T\right)|\mathcal{X}_s\right] \\ &= E\left[E\left[D_T|\mathcal{X}_T\right] F\left(X(u), u \leq T\right)|\mathcal{X}_s\right] \\ &= E\left[F\left(X(u), u \leq T\right)|\mathcal{X}_s\right]. \end{aligned} \tag{12}$$

We then get the NKP property provided we also have

$$E^Q\left[F\left(X(u), u \leq T\right)|\mathcal{F}_s\right] = E^Q\left[F\left(X(u), u \leq T\right)|\mathcal{X}_s\right] \tag{13}$$

which is precisely the condition that $\mathcal{X}$ is self sufficient in $\mathcal{F}$ under Q.

Hence we may state the result

Proposition 2 *If X is self sufficient in F under both P and Q then the NKP property holds if and only if*

$$E\left[D_T|\mathcal{X}_T\right] = 1. \tag{14}$$

These considerations motivate our interest in the property (14) as a vehicle in understanding the NKP property. In addition we present explicit results showing how self sufficiency may be maintained under a change of probability.

We may usefully decompose the full NKP property into two properties. The first we call the almost NKP property, ANKP, obtained via conditioning on the left hand side of Eq. (9) by $\mathcal{Z}_s$ in place of $\mathcal{F}_s$. The second property is then the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ under Q or Eq. (13). This is studied later in Sect. 5 where we observe that except for specially designed kernels with what we call hidden exposures to the self sufficient filtration the self sufficiency is preserved under Q and we do in fact have the NKP property. In the continuous process context these hidden exposures are stochastic integrals with respect to martingales in the self sufficient filtration that have integrands with zero conditional expectation and hence we say the exposures are hidden to the self sufficient filtration.

We thus define X to have the ANKP property with respect to D if there exists a filtration $\mathcal{Z}$ which is self sufficient in $\mathcal{F}$ under P containing $\mathcal{X}$ such that for all functionals of the paths of X , $F = F((X(u)_{0\leq u\leq T}))$ it is the case that for all $s < T$

$$E^{Q}\left[F(X(u)_{0\leq u\leq T})|\mathcal{Z}_s\right] = E^{P}\left[F(X(u)_{0\leq u\leq T})|\mathcal{Z}_s\right], \tag{15}$$

$$E^{P}\left[DF(X(u)_{0\leq u\leq T})|\mathcal{Z}_s\right] = E^{P}\left[F(X(u)_{0\leq u\leq T})|\mathcal{Z}_s\right]. \tag{16}$$

The condition (16) is equivalent to the assertion that there is no change of law of the process X on the filtration $\mathcal{Z}$. But as the probability law depends on the filtration there may be a change of law on the larger filtration $\mathcal{F}$. In addition to ANKP we require the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ under Q to lift the absence of a change of probability on $\mathcal{Z}$ to the absence of such a change of probability on $\mathcal{F}$. We further note by Proposition 1 and a computation similar to (12), that the condition $E[D_T|\mathcal{X}_T] = 1$ is both necessary and sufficient for the ANKP property in the case $\mathcal{X}$ is itself self sufficient in $\mathcal{F}$.

Finally we note that the structure of risks studied need not be associated with the continuous time paths of processes. One could be interested in other filtrations of the economy. We might consider for example just the set of times at which the stock markets attain a new peak and then extract the filtration of the economy at these times. Such a sequence of filtrations is no longer associated with the path of a process in continuous time. We may then ask whether the NKP property holds for such a filtration. In case it does, then derivatives written on the market index with payoffs measurable with respect to its peaks would be priced with no change of probability.

Hence we define more generally for an arbitrary increasing filtration $\mathcal{G} = \{\mathcal{G}_t, t \leq T\}$, that $\mathcal{G}$ is *not kernel priced* or has the NKP property with respect to D if for $s \leq T$, $G_T \geq 0$,and G_T is $\mathcal{G}_T$ measurable

$$E^{Q}\left[G_T|\mathcal{F}_s\right] = E^{P}\left[G_T|\mathcal{Z}_s\right], \tag{17}$$

where $\mathcal{Z}$ is a filtration containing $\mathcal{G}$ and $\mathcal{Z}$ is self sufficient in $\mathcal{F}$.

Similarly we define that $\mathcal{G}$ has the *almost not kernel priced property* (ANKP) with respect to D if for $s \leq T$, $G_T \geq 0$,and G_T is $\mathcal{G}_T$ measurable

$$E^{Q}\left[G_T|\mathcal{Z}_s\right] = E^{P}\left[G_T|\mathcal{Z}_s\right], \tag{18}$$

where $\mathcal{Z}$ is a filtration containing $\mathcal{G}$ and $\mathcal{Z}$ is self sufficient in $\mathcal{F}$.

Apart from the not kernel priced or NKP risks, we are also interested in the set of risks that are definitely priced. We next define the set of risks that are strongly kernel priced SKP and strongly negatively kernel priced SNKP.

We say that X is *strongly kernel priced* or has the property SKP if for all n , $s < T$ and times $t_1 < t_2 < \cdots < t_n \leq T$ and arbitrary levels $a_1, a_2, \ldots, a_n$

$$Q\left((X(t_i) < a_i)_{1\leq i\leq n}|\mathcal{F}_s\right) > P\left((X(t_i) < a_i)_{1\leq i\leq n}|\mathcal{F}_s\right). \tag{19}$$

For a strongly priced risk, in particular the conditional marginal cumulative distribution functions have been lifted up. Such a condition in the univariate case is well known to be equivalent to the statement that the law of X under P first order stochastically dominates the law under Q. Equivalently we have the property that every monotone increasing function of a strongly priced risk has a higher expectation under P than under Q and hence a positive risk premium. We call such monotone increasing functions of a risk X, risks that are aligned with X. For strongly priced risks all aligned risks have a positive risk premium.

Alternatively, we say that X is *strongly negatively kernel priced* or has the SNKP property if $-X$ has the SKP property. For strongly negatively priced risks all aligned random variables have a negative risk premium.

We first show that the density D_T itself has the SNKP property. Furthermore, the equivalence of random variables aligned with the kernel is unique in possessing this property. Hence the acquisition of cash flows aligned with the pricing kernel are the classic insurance contracts of the economy. This observation is in line with basic economic principles that indicate a relatively over weighting of states with the worse outcomes in measure changes. However, here we see these properties at a more general level without any appeal to utility theory. These are the considerations that lead to a basic representation of excess returns in terms of covariations of returns with the negative of the growth rate in the pricing density (See Back 1991).

Proposition 3 *D_T has the SNKP property.*

Proof We may observe that D_T is SNKP by first considering $n = 1, s < t \leq T$ and $a < D_s$ for which

$$\begin{aligned} Q\left[(D(t) < a)|\mathcal{F}_s\right] &= \frac{E^P\left[\mathbf{1}_{D(t)<a} D_T|\mathcal{F}_s\right]}{D_s} \\ &= \frac{E^P\left[\mathbf{1}_{D(t)<a} D_t|\mathcal{F}_s\right]}{D_s} \\ &< E^P[\mathbf{1}_{D(t)<a}|\mathcal{F}_s], \end{aligned}$$

where the last inequality follows on noting that on the set $D(t) < a$, $\frac{D(t)}{D(s)} < \frac{a}{D(s)} < 1$ by the choice of a. On the other hand for $s < t \leq T$ and $a > D(s)$ we have

$$\begin{aligned} Q\left[(D(t) > a)|\mathcal{F}_s\right] &= \frac{E^P\left[\mathbf{1}_{D(t)>a} D_T|\mathcal{F}_s\right]}{D_s} \\ &= \frac{E^P\left[\mathbf{1}_{D(t)>a} D_t|\mathcal{F}_s\right]}{D_s} \\ &> E^P[\mathbf{1}_{D(t)>a}|\mathcal{F}_s]. \end{aligned}$$

Hence all the conditional cumulative marginals distribution functions for $D(t)$ have been shifted downwards. This argument may be generalized by induction to show that D_T has the SNKP property. □

The result that the marginal cumulative distributions of D_T under Q are shifted down is related to the Hardy–Littlewood result (See Dubins and Gilat 1978) that

$$v(x) = E\left[X|X > x\right]$$

is an increasing function of x. For X a positive change of measure density the function v is unity at 0 and hence is above unity over its range. However, it is also the ratio of the probability that $(X > x)$ under Q to that under P.

Proposition 4 *D_T is the only equivalence class of random variables (up to increasing monotonic transformations) with the property that every monotone increasing function of the random variable has a negative risk premium.*

Proof Consider a random variable Γ that is not aligned with the density. Define by C the set of all random variables aligned with Γ. The set C contains all random variables that are monotone increasing functions of Γ and we easily see that C is a convex set and $D_T - 1 \notin C$. By convex set separation we may obtain a random variable Y such that

$$E[Y(D_T - 1)] < 0 \tag{20}$$

and for all $X \in C$ we have

$$E\left[YX\right] \geq 0.$$

It follows from Eq. (20) that Y has a positive risk premium. Furthermore if Y is not aligned with Γ then one may construct X aligned with Γ for which $E[XY] < 0$ and so $Y \in C$ and we have a variable aligned with Γ with a positive risk premium. □

4.1 The examples

We now investigate further the NKP properties with respect to a specific D. Our first example supports the basic intuition of considering correlations in making judgements about the kernel pricing of risks or about the NKP property.

Example 1 If $X = (X_t, 0 \leq t \leq T)$ is a $(\mathcal{F}, P)$ one dimensional Brownian motion and X, D are orthogonal as $(\mathcal{F}, P)$ martingales then X has the NKP property with $\mathcal{Z} = \mathcal{F}$.

Proof The orthogonality assumption implies that X is a Q local martingale, and since the quadratic variation of X under Q is the same as that under P by Girsanov's theorem we have that

$$\langle X\rangle_t^Q = \langle X\rangle_t^P = t,$$

and it follows by Lévy's theorem that X is a $(\mathcal{F}, Q)$ Brownian motion; in particular, (8) holds. $\square$

For a particular illustration of Example 1, let $B = (B_t, 0 \leq t \leq T)$ be a n dimensional Brownian motion with $n > 1$ and let $\mathcal{F} = \{\sigma\{B_s, s \leq t\}, t \leq T\}$. Then we know that:

$$D(t) = 1 + \int_0^t \delta_s \cdots \mathrm{d}B_s$$

for some $\mathcal{F}$ predictable process δ_s such that $\int_0^t |\delta_s|^2 \mathrm{d}s$ is almost surely finite, and that likewise:

$$X(t) = X(0) + \int_0^t x_s \cdots \mathrm{d}B_s$$

with our assumption implying here that

$$\begin{aligned} |x_s|^2 &= 1, \quad \mathrm{d}s\mathrm{d}P \text{ almost surely} \\ x_s \cdot \delta_s &= 0. \end{aligned}$$

Even more specifically we could take $n = 2, B = (U, V)$ a two dimensional standard Brownian motion and

$$\begin{aligned} D(t) &= \exp\left(A(t) - \frac{t}{2}\right) \\ A(t) &= \int_0^t \frac{U(s)\mathrm{d}U(s) + V(s)\mathrm{d}V(s)}{\sqrt{U(s)^2 + V(s)^2}} \\ X(t) &= \int_0^t \frac{-V(s)\mathrm{d}U(s) + U(s)\mathrm{d}V(s)}{\sqrt{U(s)^2 + V(s)^2}}. \end{aligned}$$

This example illustrates the type of considerations involved in widely used continuous time models supporting correlation calculations as fundamental to evaluating the kernel pricing of risk. We see here a tight connection between the absence of correlation and the absence of a change of probability or the NKP property. However, the situation can be more subtle even in the standard context of a one dimensional Brownian motion model. This is illustrated in our next example.

Example 2 We consider here $X = (X_t, t \leq T)$ as a $(\mathcal{F}\ , P)$ Brownian motion, $\mathcal{X}_T = \sigma\ \{X_t, t \leq T\}$ with the property that

$$E^P\left[D(T)|\mathcal{X}_T\right] = 1. \tag{21}$$

These constructions give us examples of the ANKP property with $\mathcal{Z} = \mathcal{X}$ under P. In fact we have noted that (21) is equivalent to (16) when $\mathcal{X}$ is self sufficient in F under P.

In particular, consider for example in a one dimensional Brownian motion $(B(t), t \leq T)$ context:

$$D(T) = 1 + \lambda \text{sign}(B(T)) \tag{22}$$

for $|\lambda| < 1$, and

$$X(t) = \int_0^t \text{sign}(B(s))\mathrm{d}B(s).$$

As is well known, the filtration of X is self sufficient (so $\mathcal{Z} = \mathcal{X}$), and $\mathcal{X}(t) = \sigma\ \{|B(s)|, s \leq t\}$, hence $\mathcal{X}(T)$ is independent from sign$(B(T))$ which immediately implies that (21) holds and hence that

$$Q_{|\mathcal{X}(T)} = P_{|\mathcal{X}(T)}. \tag{23}$$

But note that on the filtration $\mathcal{F}$, $X = (X_t, t \leq T)$ is a $(\mathcal{F}, P)$ Brownian motion and not a $(\mathcal{F}, Q)$ Brownian motion and has on $\mathcal{F}$ a nonzero instantaneous correlation structure with the kernel. It follows in particular from the considerations we investigate later in Sect. 5, that $\mathcal{X}$ is not self sufficient in $\mathcal{F}$ under Q and the ANKP does not lead to NKP in this case.

We mention this case as it is often customary to define a filtration under P describing a set of risks as for example in a stochastic volatility model of the Heston type where it is clear that the filtration describing the evolution of volatility and the stock is self sufficient under P. Now a prospective measure change could leave the law of the stock price and volatility unchanged when attention is restricted to this filtration and so a researcher may be inclined to believe that they have NKP but in fact they may only have ANKP for under Q the market price of volatility risk may be dependent on other variables that are correlated with either the stock or the volatility and these must be included in the self sufficient filtration under Q.

This example may instructively be generalized somewhat by considering an odd functional of Brownian motion, that is:

$$\Phi(B) = \Phi(B(u), u \leq T),$$

such that

$$(i) \ \ ||\Phi||_{\infty} < 1$$
$$(ii) \ \Phi(B) = -\Phi(-B).$$

Now take

$$D(T) = 1 + \Phi(B).$$

It is still true that

$$E^P[D(T)|\mathcal{X}(T)] = 1,$$

as (21) is equivalent to (24)

$$E^P\left[\Phi(B)|\mathcal{X}(T)\right] = 0, \tag{24}$$

and (24) follows from the oddity of Φ, condition (ii) above, since for every $F \geq 0$

$$E^P\left[F(|B(u)|, u \leq T)\Phi(B)\right] = -E^P\left[F(|B(u)|, u \leq T)\Phi(B)\right]$$

and hence (24).

Although this argument seems very easy, if we are given $D = (D(t), t \leq T)$ in exponential form

$$D(t) = \exp\left(\int_0^t h_s(B)\mathrm{d}B(s) - \frac{1}{2}\int_0^t h_s^2(B)\mathrm{d}s\right)$$

with $\int_0^t h_s^2(B)\mathrm{d}s < \infty$ almost surely for all $t \leq T$, then it may not be so obvious at first sight that (23) is satisfied. In fact for the example of Eq. (22) we may use the general formula for every bounded Borel function f, with $\mathcal{B}(t) = \sigma\left\{B(s), s \leq t\right\}$,

$$E\left[f(B(T))|\mathcal{B}(t)\right] = E[f(B(T))] + \int_0^t \frac{\partial V}{\partial B}|_{(s,B(s))}\mathrm{d}B(s)$$
$$V(s,x) = E\left[f(B(T))|B(s) = x\right]$$
$$= \int_{-\infty}^{\infty} \mathrm{d}y \frac{\exp\left(-\frac{(y-x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}} f(y).$$

For our example of Eq. (22) we have

$$
\begin{aligned}
V(s,x) &= 1+\lambda\int_{-\infty}^{\infty} dy\frac{\exp\left(-\frac{(y-x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}}\mathrm{sign}(y)\\
&= 1+\lambda\int_{0}^{\infty} \mathrm{d}y\frac{\exp\left(-\frac{(y-x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}} - \lambda\int_{0}^{\infty} \mathrm{d}y\frac{\exp\left(-\frac{(y+x)^2}{2(T-s)}\right)}{\sqrt{2\pi(T-s)}}\\
&= 1+\lambda\left(2N\left(\frac{x}{\sqrt{T-s}}\right)-1\right),
\end{aligned}
$$

where $N(x)$ is the standard normal distribution function. It follows that

$$
\begin{aligned}
\frac{\partial V}{\partial x} &= \frac{2\lambda}{\sqrt{(T-s)}}n\left(\frac{x}{\sqrt{T-s}}\right)\\
&\underset{\mathrm{Def}}{=} q(s,x)
\end{aligned}
$$

where $n(x)$ is the standard normal density.

To develop the exponential form for $D(t)$ we note that

$$
\begin{aligned}
D(t) &= V(t,B(t))\\
&= 1+\int_0^t q(s,B(s))\mathrm{d}B(s)\\
&= 1+\int_0^t V(s,B(s))\left(\frac{q(s,B(s))}{V(s,B(s))}\right)\mathrm{d}B(s)\\
&= \exp\left(\int_0^t\left(\frac{q(s,B(s))}{V(s,B(s))}\right)\mathrm{d}B(s)-\frac{1}{2}\int_0^t\left(\frac{q(s,B(s))}{V(s,B(s))}\right)^2\mathrm{d}s\right)
\end{aligned}
$$

and we have

$$
\begin{aligned}
&h(s,B(s))\\
&\quad=\left(1+\lambda\left(2N\left(\frac{B(s)}{\sqrt{T-s}}\right)-1\right)\right)^{-1}\frac{2\lambda}{\sqrt{(T-s)}}n\left(\frac{B(s)}{\sqrt{T-s}}\right).
\end{aligned}
$$

In general we have

$$
\Phi(B)=\int_0^T \phi(s,(B(u),u\le s))\mathrm{d}B(s).
$$

From the oddity of Φ and the uniqueness of ϕ we deduce that ϕ is even or that

$$\phi(s,(B(u),u\leq s))=\phi(s,-(B(u),u\leq s))$$

as is also the case for our example $h(s,B(s))$. Under Q we get

$$\begin{aligned}B(t)&=\widetilde{B}(t)+\int_0^t\frac{\phi(s,(B(u),u\leq s))}{1+\Phi(s,(B(u),u\leq s))}\mathrm{d}s\\&=\widetilde{B}(t)+\int_0^t\frac{\phi(s,(B(u),u\leq s))}{1+E\left[\Phi(B)|\mathcal{F}_s\right]}\mathrm{d}s\end{aligned}$$

with $\widetilde{B}$ an $(\mathcal{F},Q)$ Brownian motion. We may now exhibit the dynamics of X under Q,

$$\begin{aligned}X(t)&=\int_0^t\operatorname{sign}(B(s))\mathrm{d}B(s)\\&=\int_0^t\operatorname{sign}(B(s))\mathrm{d}\widetilde{B}(s)+\int_0^t\frac{\operatorname{sign}(B(s))\phi(s,(B(u),u\leq s))}{1+\Phi(s,(B(u),u\leq s))}\mathrm{d}s\end{aligned}$$

so that $X(t)$ is a semimartingale under Q reflecting a nonzero correlation structure but $X(t)$ has the ANKP property. That $X(t)$ is a martingale with respect to its own filtration $\mathcal{X}(t)$ follows from the fact that

$$E^Q\left[\frac{\operatorname{sign}(B(s))\phi(s,(B(u),u\leq s))}{1+\Phi(s,(B(u),u\leq s))}|\mathcal{X}(s)\right]=0. \tag{25}$$

This follows on observing that the left hand side of Eq. (25) is

$$\begin{aligned}\frac{E^P\left[D(s)\operatorname{sign}(B(s))|\mathcal{X}(s)\right]}{E^P\left[D(s)|\mathcal{X}(s)\right]}&=E^P\left[\operatorname{sign}(B(s))|\mathcal{X}(s)\right]\\&=0.\end{aligned}$$

Example 2 illustrates an interesting situation: The existence of a change of measure density that prices positively the risk of the Brownian motion driving the economy continuously through time: Yet there exist martingales correlated with the change of measure density continuously through time that are not priced when attention is restricted to the risk filtration itself. It is true that in this example the expected correlation projected onto the filtration $\mathcal{X}(s)$ of the martingale is zero by construction, but in practice such a situation is not that easy to detect.

We grant that Example 2 is one dimensional but a comparable discussion may easily be embedded into a higher dimensional context. For example we could consider the Heston model for the evolution of the stock price $S(t)$ and the volatility $v(t)$. Now consider a pricing kernel of the form

$$D(T) = 1 + \lambda \mathrm{sign}(B(T))F(S(T), v(T)) \tag{26}$$

for a bounded and positive function $F(S(T), v(T))$ with $B(t), t > 0$ an independent Brownian motion. Let $\mathcal{X}$ be the filtration of the stock and its volatility. For any $\mathcal{X}_T$ measurable random variable C_T we have that

$$\begin{aligned} E^P\left[C_T D(T)\right] &= E^P[E^P\left[C_T D(T)|\mathcal{X}_T\right]] \\ &= E^P[C_T E^P[D(T)|\mathcal{X}_T]] \\ &= E^P[C_T]E^P[D(T)] \end{aligned}$$

and we have no correlation between the risk and the density as random variables at time zero. We also have no correlation between the risk and the density as random variables at all times $s > 0$ with respect to the risk filtration as

$$\begin{aligned} E^P\left[C_T D(T)|\mathcal{X}_s\right] &= E^P[E^P\left[C_T D(T)|\mathcal{X}_T\right]|\mathcal{X}_s] \\ &= E^P[C_T E^P[D(T)|\mathcal{X}_T]|\mathcal{X}_s] \\ &= E^P[C_T|\mathcal{X}_s]E^P[D(T)|\mathcal{X}_s]. \end{aligned}$$

But the filtration $\mathcal{X}$ is not self sufficient in $\mathcal{F}$ under Q and we do have correlation between the risk and the density at time $s > 0$ with respect to the filtration $\mathcal{F}_s$ as

$$E\left[C_T D(T)|\mathcal{F}_s\right] = E^P\left[E^P\left[C_T\left(1+\lambda\left(2N\left(\frac{B(s)}{\sqrt{T-s}}\right)-1\right)F(S(T),v(T))\right)\right]\middle|\mathcal{F}_s\right],$$

where the expression in brackets multiplying λ is just the conditional expectation of $\mathrm{sign}(B(T))$ conditional on $B(s)$. We could now take by way of example for C_T the variable $F(S(T), v(T))$ to get correlation conditional on $\mathcal{F}_s$. Here NKP fails but ANKP holds. By construction we have given the density an exposure to movements in the stock and volatility process that is not visible to the risk filtration and projects to zero.

To see the hidden nature of the exposure in the density (26) one explicitly writes out the $(\mathcal{F}, P)$ martingale associated with the density (26) as a terminal random variable as a stochastic integral with respect to the Brownian motions $(B(t), t > 0)$, and the two Brownian motions $(W_1(t), t > 0)$ and $(W_2(t), t > 0)$ that drive the stock and the volatility processes. Assuming that one has written

$F(S(T), v(T))$ in the form

$$F\left(S(T), v(T)\right) = E\left[F(S(T), v(T))\right] + \int_0^T \left(f_1(s)dW_1(s) + f_2(s)\mathrm{d}W_2(s)\right)$$

for some integrands f_1, f_2 one may show that the exposure of the density to the filtration generated by the stock and volatility processes takes the form

$$\begin{aligned} D(T) &= 1 + \int_0^T a(s)\mathrm{d}B(s) + \int_0^T E\left[\lambda \mathrm{sign}(B(T)|\mathcal{B}_s\right]\left((f_1(s)\mathrm{d}W_1(s) + f_2(s)\mathrm{d}W_2(s))\right) \\ &= 1 + \int_0^T a(s)\mathrm{d}B(s) + \int_0^T \lambda\left(2N\left(\frac{B(s)}{\sqrt{T-s}}\right) - 1\right)\left((f_1(s)\mathrm{d}W_1(s) + f_2(s)\mathrm{d}W_2(s))\right) \end{aligned}$$

but as the function

$$2N\left(\frac{B(s)}{\sqrt{T-s}}\right) - 1$$

is an antisymmetric function of $B(s)$ that is independent of the stock and volatility processes, its projection onto this filtration is zero. The exposure of the density to the stock and volatility filtration is therefore hidden from this filtration.

Our next example illustrates the possibility in one dimension of a situation where we have zero correlation between a risk and the change of measure but the risk in question is kernel priced and does not have the NKP property.

Example 3 Define the density process by

$$D(t) = \exp\left(\int_0^t \mathbf{1}_{B(s)>0}\mathrm{d}B(s) - \frac{1}{2}\int_0^t \mathbf{1}_{B(s)>0}\mathrm{d}s\right)$$

and let $X = (X(t), t \leq T)$ be the martingale

$$X(t) = \int_0^t \mathbf{1}_{B(s)<0}\mathrm{d}B(s).$$

The law of X is changed by the density D.

Proof It is clear by construction that

$$\begin{aligned}\langle X, D\rangle_t &= \int_0^t D(s)\mathbf{1}_{B(s)>0}\mathbf{1}_{B(s)<0}\mathrm{d}s \\ &= 0\end{aligned}$$

and there is no correlation between the density D and the martingale X.

However the quadratic variation of X is

$$\begin{aligned}\langle X\rangle_t &= \int_0^t \mathbf{1}_{B(s)<0}\mathrm{d}s \\ &= t - \int_0^t \mathbf{1}_{B(s)>0}\mathrm{d}s\end{aligned}$$

and the law of this quadratic variation under Q is different from its law under P. More specifically:

$$E^Q\left[\mathbf{1}_{(B(t)<0)}\right] < \frac{1}{2} = E^P\left[\mathbf{1}_{(B(t)<0)}\right].$$

Indeed

$$\begin{aligned}E^Q\left[\mathbf{1}_{(B(t)<0)}\right] &= E^P\left[\mathbf{1}_{(B(t)<0)}D(t)\right] \\ &= E^P\left[\mathbf{1}_{(B(t)<0)}\exp\left(-\frac{1}{2}\Sigma(t) - \frac{1}{2}\int_0^t \mathbf{1}_{(B(s)>0)}\mathrm{d}s\right)\right]\end{aligned}$$

since on $(B(t) < 0), L(t) = \int_0^t \mathbf{1}_{B(s)>0}\mathrm{d}B(s) \equiv B(t)^+ - \frac{1}{2}\Sigma(t) = -\frac{1}{2}\Sigma(t)$ where $\Sigma(t)$ is the local time of Brownian motion at zero. □

Another example of zero correlation with absence of NKP property is provided by the following two dimensional construction.

Example 4 Let $\mathbb{B} = (U, V)$ be a standard two dimensional Brownian motion. Define

$$\begin{aligned}R^2(t) &= U^2(t) + V^2(t), \\ D(t) &= \exp\left(\int_0^t U(s)\mathrm{d}U(s) + V(s)\mathrm{d}V(s) - \frac{1}{2}\int_0^t R^2(s)\mathrm{d}s\right)\end{aligned}$$

$$= \exp\left(\frac{1}{2}(R^2(t) - 2t) - \frac{1}{2}\int_0^t R^2(s)\mathrm{d}s\right).$$

Further let

$$X(t) = \int_0^t -U(s)\mathrm{d}V(s) + V(s)\mathrm{d}U(s)$$

then X is a martingale under both P and Q but X does not have the NKP property.

Proof We easily verify that X remains a martingale under Q as

$$\begin{aligned}\langle X, D\rangle_t &= \int_0^t D(s)(-U(s)V(s) + U(s)V(s))\mathrm{d}s \\ &= 0.\end{aligned}$$

But the probability law of X under Q is not that under P. In fact both under P and Q we have that

$$X(t) = \gamma\left(\int_0^t R^2(s)\mathrm{d}s\right),$$

where $(\gamma(u), u \geq 0)$ is a one dimensional Brownian motion independent of R, but the law of R (hence the law of $\langle X\rangle$) is different under P and Q.

Under Q we have that

$$U(t) = \widetilde{U}(t) + \int_0^t U(s)\mathrm{d}s,$$

$$V(t) = \widetilde{V}(t) + \int_0^t V(s)\mathrm{d}s,$$

where $\left(\widetilde{U}, \widetilde{V}\right)$ is a two dimensional $(\mathcal{F}, Q)$ Brownian motion; hence, under Q, U and V are two independent Ornstein–Uhlenbeck processes, and

$$R^2(t) = 2\int_0^t R(s)\mathrm{d}\widetilde{\beta}(s) + 2\int_0^t R^2(s)\mathrm{d}s + 2t$$

with $\widetilde{\beta} = (\widetilde{\beta}(t), t \geq 0)$ a Brownian motion, is a *CIR* (Cox et al. 1985) type process. While under P we have that

$$R^2(t) = 2\int_0^t R(s)\mathrm{d}\beta(s) + 2t,$$

for a Brownian motion $\beta = (\beta(t), t \geq 0)$. We note that under the respective probabilities P and Q, the Brownian motions $\widetilde{\beta}, \beta$ are independent of γ . □

We have now seen examples of nonzero correlation and presence of the ANKP property and examples of zero correlation and absence of the NKP property. From the perspective of financial economics and valuation of claims, the NKP property is fundamental. An analysis of the ANKP property in terms of orthogonality is precisely developed in Sect. 4.2. This is followed in Sect. 5 by a study of the invariance of self sufficiency to the measure change, the property needed to infer NKP from ANKP. We now illustrate the case of a risk satisfying ANKP with respect to the simplest measure change in the one-dimensional Brownian context.

Example 5 Consider the change of measure density

$$D(t) = \exp\left(\lambda B(t) - \frac{\lambda^2 t}{2}\right)$$

so that under Q, $B(t)$ is a Brownian motion with drift λ. Let

$$X(t) = B(t) - \int_0^t \frac{B(s)}{s}\mathrm{d}s.$$

The semimartingale $X = (X(t), t \geq 0)$ is a Brownian motion both under P and Q.

Proof Let $\mathcal{G}(t) = \sigma\{X(u), u \leq t\}$. We verify that in its own filtration the process X is Brownian motion. It is clear that it is a Gaussian process with zero mean and covariance function

$$C(s,t) = E^P\left[\left(B(t) - \int_0^t \frac{B(u)}{u}\mathrm{d}u\right)\left(B(s) - \int_0^s \frac{B(v)}{v}\mathrm{d}v\right)\right]$$

and for $s < t$ we have

$$
\begin{aligned}
C(s,t) &= s - E^P\left[\int_0^s \frac{B(u)B(s)}{u}\mathrm{d}u + \int_s^t \frac{B(u)B(s)}{u}\mathrm{d}u\right] - E^P\left[\int_0^s \frac{B(t)B(v)}{v}\mathrm{d}v\right] \\
&\quad + E^P\left[\int_0^s\int_0^s \frac{B(u)B(v)}{uv}\mathrm{d}v\mathrm{d}u + \int_s^t\int_0^s \frac{B(u)B(v)}{uv}\mathrm{d}v\mathrm{d}u\right] \\
&= s - s - s\log\left(\frac{t}{s}\right) - s + \int_0^s\int_0^s \frac{u\wedge v}{uv}\mathrm{d}v\mathrm{d}u + s\log\left(\frac{t}{s}\right) \\
&= -s + \int_0^s \mathrm{d}u \int_0^u \mathrm{d}v\frac{1}{u} + \int_0^s \mathrm{d}u\int_u^s \frac{1}{v}\mathrm{d}v \\
&= -s + s + \int_0^s \mathrm{d}v\int_0^v \frac{1}{v}\mathrm{d}u \\
&= s.
\end{aligned}
$$

We therefore have that

$$E\left[D(t)|\mathcal{G}(t)\right] = 1.$$

In fact we have that

$$E[B(t)X(s)] = 0$$

for $s \leq t$ and $B(t)$ is independent of $\mathcal{G}(t)$. □

We see here that even the simplest measure change that prices the Brownian motion at a constant price continuously through time, fails to change probability when restricted to the filtration of the risk for some semimartingale risks. We note however, that though probability has not been changed we do not here have the NKP property as the filtration of X is not self sufficient in $\mathcal{F}$ under either P or Q.

The result that $X(t)$ in Example 5 is a Brownian motion is explained in greater detail in Chapter 1 of Yor (1992). It is obtained by considering the semimartingale decomposition of Brownian motion $(B(u), u \leq t)$ enlarged by its value $B(t)$ at time t, and then reversing time from time t.

We further comment that $X(t)$ in Example 5 has the ANKP property with respect to any change of measure density $D(t) = h(B(t), t)$ where h is a positive space time harmonic function which is known to be of the form

$$h(B(t),t) = \int_{-\infty}^{\infty} \mu(\mathrm{d}\lambda) \exp\left(\lambda B(t) - \frac{\lambda^2 t}{2}\right)$$

for any probability measure μ on $\mathbb{R}$.

4.2 Almost no kernel pricing for filtrations

In this subsection we clarify precisely what the ANKP property means in terms of orthogonality relations between certain martingales. For this purpose we discuss the ANKP property just at the level of filtrations as defined in (18). We suppose that $\mathcal{G}$ is the filtration under consideration and $\mathcal{Z}$ is a filtration containing $\mathcal{G}$ and self sufficient in $\mathcal{F}$.

Proposition 5 *$\mathcal{G}$ has the ANKP property with respect to D if and only if every $(\mathcal{Z}, P)$ martingale whose terminal value is $\mathcal{G}_T$ measurable is orthogonal to the martingale $E^P[D_T|\mathcal{Z}_t]$.*

Proof Let $M(t)$ be a $(\mathcal{Z}, P)$ martingale with terminal value G_T, which is $\mathcal{G}_T$ measurable. By the ANKP property we may write

$$E^Q\left[G_T|\mathcal{Z}_s\right] = E^P\left[G_T|\mathcal{Z}_s\right].$$

The left hand side is given by

$$E^Q\left[G_T|\mathcal{Z}_s\right] = \frac{E^P\left[D_T G_T|\mathcal{Z}_s\right]}{E^P\left[D_T|\mathcal{Z}_s\right]}.$$

Substituting above we get that

$$E^P\left[D_T G_T|\mathcal{Z}_s\right] = E^P\left[D_T|\mathcal{Z}_s\right] E^P\left[G_T|\mathcal{Z}_s\right].$$

and the result holds. □

Proposition 5 describes the precise link between orthogonality and the ANKP property. For example, it is not sufficient to just test the orthogonality of X and D in the case $\mathcal{G}$ is generated by the process X. One needs to consider the martingales in $(\mathcal{Z}, P)$ with $\mathcal{G}_T$ measurable terminal values and ensure that these are all orthogonal to $E^P[D_T|\mathcal{Z}_t]$. Typically the filtration $\mathcal{Z}$ would contain variables necessary for predicting outcomes in $\mathcal{G}$ and the law of these variables may be altered by the change of probability and hence indirectly the law of $\mathcal{G}$ measurable variables. Hence orthogonality tests need to reach beyond just X and D to confirm the ANKP result. However, in many cases it is sufficient to simply check that $E^P[D_T|\mathcal{Z}_t]$ is trivial. This happens because the collection of $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_T$ often generates all the $(\mathcal{Z}, P)$ martingales in the sense of Kunita–Watanabe.

In the special case of $\mathcal{G}$ itself being self sufficient in $\mathcal{F}$ we choose systematically that $\mathcal{Z} = \mathcal{G}$ and we need only consider all the $\mathcal{G}$ martingales in this case. In addition when $\mathcal{G}$ has the martingale representation property with respect to a set of martingales $(\gamma_j, j \in J)$ then one may restrict the orthogonality tests to just these martingales. It follows that if $\mathcal{G}$ has the martingale representation property with respect to X , which is both a $\mathcal{G}$ and $\mathcal{F}$ martingale, then $\mathcal{G}$ is self sufficient in $\mathcal{F}$ and a mere test of the orthogonality of X and D suffices.

It is now helpful to construct a typical situation resulting in the ANKP property for some risk. For this we wish to take for $\mathcal{G}$ the filtration of a risk not self sufficient in $\mathcal{F}$ under P. This means that $\mathcal{F}_s$ contains information about risks in $\mathcal{G}_T$ that $\mathcal{G}_s$ does not have. One of the simplest examples for such a relation between filtrations is that of Brownian bridges whereby we take for $\mathcal{G}$ the filtration of a Brownian motion, $(B_u, u > 0)$, with $\mathcal{G}_t = \sigma(B_u, u \leq t)$ and $\mathcal{G} = \{\mathcal{G}_t, t \leq S\}$. For $T > S$ and for another independent Brownian motion $(W_u, u > 0)$ we let $\mathcal{F}_t = \{\sigma(B(T)) \vee \sigma(B_u, u \leq t) \vee \sigma(W_u, u \leq t)\}$, with $\mathcal{F} = \{\mathcal{F}_t, t \leq S\}$. For the filtration $\mathcal{Z} \supset \mathcal{G}$ self sufficient in $\mathcal{F}$ we take $\mathcal{Z}_t = \{\sigma(B(T) \vee \sigma(B_u, u \leq t)\}$ with $\mathcal{Z} = \{\mathcal{Z}_t, t \leq S\}$.

We now choose the density D to be $\mathcal{F}_S$ measurable with a projection on $\mathcal{Z}$ that is orthogonal to all the $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$. For this it is important to know what these martingales are and what are the $(\mathcal{Z}, P)$ martingales orthogonal to all the $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$, if any. We now recognize that in this case $\mathcal{Z}_S = \mathcal{G}_S \vee \sigma(B(T) - B(S))$ and $B(T) - B(S)$ is independent of $\mathcal{G}_S$. Hence the projection of D onto $\mathcal{Z}_S$ being $B(T) - B(S)$ measurable will give the required orthogonality of the projection to all the $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$. This will deliver the ANKP property of $\mathcal{G}_S$ with respect to D. In particular we may take

$$D = \exp\left(B(T) - B(S) - \frac{T-S}{2}\right) \times \mathcal{E}\left(\int_0^S v(s)\mathrm{d}B(s) + w(s)\mathrm{d}W(s)\right) \tag{27}$$

for any processes $v(s), w(s)$ that are adapted to the $\mathcal{F}$ filtration. The projection of D on $\mathcal{Z}_S$ is then the $\mathcal{Z}_S$ measurable random variable

$$\Delta = \exp\left(B(T) - B(S) - \frac{T-S}{2}\right)$$

whose associated $(\mathcal{Z}, P)$ martingale is orthogonal to all $(\mathcal{Z}, P)$ martingales finishing in $\mathcal{G}_S$ and we therefore have ANKP. For the NKP property we take up this example again after we have established the conditions for NKP.

4.3 ANKP for filtrations and the examples

We now reexamine the five examples in the light of Proposition 5. For the first example we note that $\mathcal{G}$ is the filtration generated by the Brownian motion X.

$\mathcal{G}$ has the martingale representation property with respect to X hence is self sufficient in $\mathcal{F}$ and one only needs to verify the orthogonality of D with respect to X. This example is in line with our final comment in Sect. 4.2.

In the statement of Example 1 we can relax the hypothesis that X is a one dimensional Brownian motion and take for X any extremal martingale, which is a martingale for which there is only one equivalent martingale measure with respect to its own filtration $\mathcal{X}$.

In Example 2 $\mathcal{G}$ is self sufficient and hence $\mathcal{Z}$ is chosen equal to $\mathcal{G}$. Furthermore $E^P[D_T|\mathcal{G}_t]$ is trivially orthogonal to all the $\mathcal{G}$ martingales and hence we have the ANKP property. We comment for Example 3, that the NKP property does not hold and the filtration of X is not self sufficient in $\mathcal{F}$. Indeed there are discontinuous martingales in the $\mathcal{X}$ filtration, Lane (1978) and Knight (1987). In contrast, Example 4 is comparable to three but the filtration of X is that of $\mathcal{F}$ when the two Brownian motions start away from zero. Example 5 does not quite fit the structure of Proposition 5 as the filtration of X is not self sufficient in $\mathcal{F}$.

4.4 A decomposition of risks

In the context of self sufficient filtrations we ask if it is possible to decompose all risks viewed as $\mathcal{F}$ measurable martingales as the sum of two risk classes, those that are not kernel priced with a restricted filtration $\mathcal{G}$ and have the ANKP property, and those that are kernel priced with the restricted filtration $\mathcal{H}$. A more precise statement follows below.

We say that two filtrations $\mathcal{G}$ and $\mathcal{H}$ *supplement* each other in $\mathcal{F}$ if both are self sufficient, every $\mathcal{G}$ martingale is orthogonal to every $\mathcal{H}$ martingale, and every square integrable $\mathcal{F}$ martingale is generated in the sense of Kunita and Watanabe by the square integrable martingales with respect to either $\mathcal{G}$ or $\mathcal{H}$.

Furthermore we say that a *pricing decomposition* of $\mathcal{F}$ risks with respect to D exists if there exist two filtrations $\mathcal{G}$ and $\mathcal{H}$ supplementing each other in $\mathcal{F}$ with the further requirement that every $\mathcal{G}$ martingale has the ANKP property and no nontrivial $\mathcal{H}$ martingale has the ANKP property.

We present some partial results in this direction at the level of square integrable martingales. We begin by selecting $\mathcal{G}^*$ a maximal self sufficient filtration with the ANKP property. We then define by $\mathcal{M}_{\mathcal{G}^*}$ the collection of all square integrable $\mathcal{G}^*$ martingales. We construct from $\mathcal{M}_{\mathcal{G}^*}$ the $\mathcal{F}$ stable subspace $\mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*})$ generated by these martingales in $\mathcal{M}_{\mathcal{F}}$. We further observe that $\mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*})$ is orthogonal to the change of measure D. Hence the orthogonal complement of $\mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*})$ in the Hilbert space $\mathcal{M}_{\mathcal{F}}$ is nonempty. We may therefore decompose the Hilbert space $\mathcal{M}_{\mathcal{F}}$ as

$$\mathcal{M}_{\mathcal{F}} = \mathcal{S}_2(\mathcal{M}_{\mathcal{G}^*}) \oplus \mathcal{K}.$$

It remains to relate $\mathcal{K}$ to the stable space generated by a set of priced risks. A more constructive characterization of the subspace of priced risks is left for future research at this stage.

5 Self sufficiency and measure changes

In this section we study conditions on the change of measure density with the property that self sufficient filtrations under P remain self sufficient under Q. As we have noted in Proposition 2, this property is essential to deliver the NKP property from the ANKP property that we have studied in detail. For simplicity we assume throughout that all $(\mathcal{F}, P)$ martingales are continuous. We introduce a collection of processes associated with the change of measure density D and the filtration $\mathcal{J}$ which we suppose is self sufficient under P in $\mathcal{F}$.

The first process we identify is the projection of D onto the filtration $\mathcal{J}$ that we define by Δ,

$$\Delta(t) = E^P\left[D(t)|\mathcal{J}_t\right].$$

We next define the stochastic logarithms of D, Δ by L, Λ as

$$L(t) = \int_0^t \frac{\mathrm{d}D(s)}{D(s)}$$

$$\Lambda(t) = \int_0^t \frac{\mathrm{d}\Delta(s)}{\Delta(s)}$$

so that $D(t) = \mathcal{E}(L)_t$, $\Delta(t) = \mathcal{E}(\Lambda)_t$.

Proposition 6 *Assume that $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P, then $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under Q if and only if the $(\mathcal{F}, P)$ local martingale*

$$L - \Lambda$$

is orthogonal to the set of all $(\mathcal{J}, P)$ local martingales. Equivalently the Kunita–Watanabe projection of L on $(\mathcal{J}, P)$ is Λ.

Proof We simultaneously establish both directions. A classical reinforcement of Girsanov's theorem states that every $(\mathcal{J}, Q)$ local martingale $\widetilde{I}^Q$ may be obtained as

$$\widetilde{I}^Q(t) = I^P(t) - \int_0^t \frac{\mathrm{d}\langle I^P, \Delta\rangle_u}{\Delta(u)} \tag{28}$$

for I^P a generic $(\mathcal{J}, P)$ local martingale.

Since under self sufficiency of $\mathcal{J}$ in $\mathcal{F}$ under P, I^P is also a $(\mathcal{F}, P)$ local martingale we may write, by using Girsanov's theorem with respect to the filtration $\mathcal{F}$ that

$$I^P(t) = J^Q(t) + \int_0^t \frac{\mathrm{d}\langle I^P, D\rangle_u}{D(u)}, \tag{29}$$

where $(J^Q(t), t \geq 0)$ is a $(\mathcal{F}, Q)$ local martingale.

Combining (28) and (29) we obtain

$$\widetilde{I}^Q(t) = J^Q(t) + \int_0^t \frac{\mathrm{d}\langle I^P, D\rangle_u}{D(u)} - \int_0^t \frac{\mathrm{d}\langle I^P, \Delta\rangle_u}{\Delta(u)}$$
$$= J^Q(t) + \int_0^t \mathrm{d}\langle I^P, L\rangle_u - \int_0^t \mathrm{d}\langle I^P, \Lambda\rangle_u.$$

Thus $\widetilde{I}^Q(t)$ is a $(\mathcal{F}, Q)$ local martingale if and only if

$$\int_0^t \mathrm{d}\langle I^P, L\rangle_u - \int_0^t \mathrm{d}\langle I^P, \Lambda\rangle_u = 0$$

which proves the result. □

Two applications of Proposition 5, in the one and two dimensional cases illustrate the type of measure changes preserving self sufficiency of a filtration.

Example 6 Assume $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that:

· every $(\mathcal{F}, P)$ martingale is a stochastic integral with respect to $(B(t), t \geq 0)$ a one dimensional Brownian motion

· every $(\mathcal{J}, P)$ martingale is a stochastic integral with respect to $(\beta(t), t \geq 0)$ a one dimensional $(\mathcal{J}, P)$ Brownian motion

Then, $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under Q if and only if

$$D = \Delta \text{ or equivalently } L = \Lambda.$$

Proof Since $L - \Lambda$ is orthogonal to all the $(\mathcal{J}, P)$ martingales and since under the joint three hypotheses these martingales generate in the sense of Kunita, all the $(\mathcal{F}, P)$ martingales it must be that $L - \Lambda = 0$. □

A concrete example is given by $\mathcal{F}$ the natural filtration of a one dimensional Brownian motion $(B(t), t \geq 0)$, and $\mathcal{J}$ the filtration of the absolute value of $(|B(t)|, t \geq 0)$. It is well known that $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that the $(\mathcal{J}, P)$ Brownian motion may be chosen such as

$$\beta(t) = \int_0^t \mathrm{sign}(B(s))\mathrm{d}B(s).$$

If we take for D

$$D(t) = \exp\left(\mu B(t) - \frac{\mu^2 t}{2}\right)$$

then $\mathcal{J}$ is not self sufficient in $\mathcal{F}$ under Q. Here we must have the density adapted to $\mathcal{J}$ for stability of self sufficiency under a change of measure. It is not so adapted. More precisely we may evaluate L and Λ as follows. First we obtain that

$$\begin{aligned}\Delta(t) &= \cosh(\mu B(t))\mathrm{e}^{-\frac{\mu^2 t}{2}} \\ &= \exp\left(\log\left(\cosh\left(\mu B(t)\right)\right) - \frac{\mu^2 t}{2}\right).\end{aligned}$$

Thus

$$\begin{aligned}L(t) &= \mu B(t) \\ \Lambda(t) &= \mu \int_0^t \tanh\left(\mu B(s)\right) \mathrm{d}B(s) \\ &= \mu \int_0^t \tanh(\mu|B(s)|)\mathrm{d}\beta(s)\end{aligned}$$

and we explicitly see that $L \neq \Lambda$.

Example 7 Assume $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that $\cdot$ every $(\mathcal{J}, P)$ martingale is a stochastic integral with respect to $(\beta(t), t \geq 0)$ a one dimensional $(\mathcal{J}, P)$ Brownian motion $\cdot$ every $(\mathcal{F}, P)$ martingale is a stochastic integral with respect $(B(t), t \geq 0)$ a two dimensional $(\mathcal{F}, P)$ Brownian motion $(\beta(t), \gamma(t)); t \geq 0$, Then, $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under Q if and only if for some $\mathcal{F}$ predictable process $l_2(s)$

$$L(t) - \Lambda(t) = \int_0^t l_2(s)\mathrm{d}\gamma(s)$$

for some $(l_2(s), s \geq 0)$ a $\mathcal{F}$ predictable process such that

$$\int_0^t (l_2(s))^2 \,\mathrm{d}s < \infty$$

and in this case we have

$$\Lambda(t) = \int_0^t \lambda(s)\mathrm{d}\beta(s) \text{ and}$$

$$L(t) = \int_0^t \lambda(s)\mathrm{d}\beta(s) + l_2(s)\mathrm{d}\gamma(s) \tag{30}$$

Proof The representation for $L(t)$ in the form

$$L(t) = \int_0^t l_1(s)\mathrm{d}\beta(s) + \int_0^t l_2(s)\mathrm{d}\gamma(s)$$

is immediate. In order that $L - \Lambda$ is orthogonal to the set of all $(\mathcal{J}, P)$ martingales it is necessary and sufficient that

$$\left\langle L - \Lambda, \int_0^t (l_1(s) - \lambda(s))\, \mathrm{d}\beta(s) \right\rangle = 0$$

or that $l_1(s) = \lambda(s)$ $\mathrm{d}s\mathrm{d}P$ almost surely. □

For a concrete example we take for $\mathcal{F}$ the natural filtration of $(U(t), V(t); t \geq 0)$ a two dimensional Brownian motion and $\mathcal{J} = \{\mathcal{J}_t : t \geq 0\}$ the filtration of

$$\left(R^2(t) = U^2(t) + V^2(t), t \geq 0\right)$$

Again it is well known that $\mathcal{J}$ is self sufficient in $\mathcal{F}$ under P and that the Brownian motion

$$\beta(t) = \int_0^t \frac{U(s)\mathrm{d}U(s) + V(s)\mathrm{d}V(s)}{R(s)}, \quad t \geq 0$$

generates all the $(\mathcal{J}, P)$ martingales. We may also take the second Brownian motion

$$\gamma(t) = \int_0^t \frac{V(s)\mathrm{d}U(s) - U(s)\mathrm{d}V(s)}{R(s)}$$

and it is well known that β and γ, which is independent of β, generate all the $(\mathcal{F}, P)$ martingales.

Consider for D

$$D(t) = \exp\left(a \int_0^t R(s)\mathrm{d}\gamma(s) - \frac{a^2}{2}\int_0^t R^2(s)\mathrm{d}s\right)$$

so that now we have

$$L(t) = a \int_0^t R(s)\mathrm{d}\gamma(s)$$
$$\Delta(t) = E^P[D(t)|\mathcal{J}_t] \equiv 1$$

or that $\Lambda(t) \equiv 0$, and $L(t) = L(t) - \Lambda(t)$ is orthogonal to the $(\mathcal{J}, P)$ martingales. Hence we still have $\mathcal{J}$ self sufficient in $\mathcal{F}$ under Q.

Combining the lessons of Examples 6 and 7 we learn that to preserve self sufficiency, the change of measure density factors as

$$D(t) = \Delta(t)\mathcal{E}(L - \Lambda)_t.$$

where $L - \Lambda$ is orthogonal to all the $(\mathcal{J}, P)$ martingales and in particular to Λ.

We consider now the density (27) in our example for ANKP using the Brownian bridge. We see that for the orthogonality of $L-\Lambda$ to all the $(\mathcal{Z}, P)$ martingales we cannot have the first integral with respect to the Brownian motion $B(s)$ and to preserve the self sufficiency of $\mathcal{Z}$ in $\mathcal{F}$ under Q we must take the density in the form

$$D = \exp\left(B(T) - B(S) - \frac{T-S}{2}\right) \times \mathcal{E}\left(\int_0^S w(s)\mathrm{d}W(s)\right)$$

with $w(s)$ adapted to the $\mathcal{F}$ filtration. Financially we have that the risk of the Brownian motion $B(u), u \leq S$ is not priced. An independent component in $\mathcal{Z}$ is priced as is the Brownian motion $W(u), u \leq S$ with the prices being $\mathcal{F}$ adapted. In this case we also have NKP.

To illustrate what is necessary for the failure of the self sufficiency of $\mathcal{J}$ in $\mathcal{F}$ under Q we observe that, in the context of Example 7, using Proposition 6,

$$L(t) - \Lambda(t) = \int_0^t l_1(s)\mathrm{d}\beta(s) + \int_0^t l_2(s)\mathrm{d}\gamma(s)$$

for some particular integrand $l_1(s)$. Indeed, since the projection onto the filtration generated by β of the integral of l_1 with respect to β is zero it must be the case that the integrand is special and in fact

$$E\left[l_1(s)|\mathcal{B}_\infty\right] = 0.$$

To establish this result let

$$M_t^u = \int_0^t u_s \mathrm{d}\beta_s$$

for any u_s predictable with respect to $\mathcal{F}$ and $E\left[\int_0^t u_s^2 \mathrm{d}s < \infty\right]$. It is a simple consequence of the martingale representation property for $(\mathcal{J}, P)$ that

$$E\left[M_t^u | \mathcal{J}_t\right] = \int_0^t E\left[u_s | \mathcal{J}_s\right] \mathrm{d}\beta_s$$

consequently $E\left[M_t^u | \mathcal{J}_t\right] = 0$ if and only if $E\left[u_s | \mathcal{J}_\infty\right] = 0$.

The density D has in this case a sort of hidden exposure to the risks of filtration $\mathcal{J}$ in that conditional on all of $\mathcal{J}_\infty$ these exposures cannot be observed by evaluating conditional expectations of the integrand of exposure. The density would have to be specially constructed with respect to the particular risk filtration to mask the conditional expectation of the exposure. Barring the presence of such hidden exposures we expect to maintain self sufficiency of $\mathcal{J}$ in $\mathcal{F}$ under Q when we have it under P.

6 Event risk and no kernel price

One is often interested in the pricing of events like default or first passage to various boundaries. In fact we now have specific event risks with a growing market like the credit default swap contracts. In this section we inquire about the no price property for events and the martingales associated with them via conditional expectations. Consider a set $A \in \mathcal{F}_T$ and the associated $(\mathcal{F}, P)$ martingale

$$X^A(t) = E^P\left[\mathbf{1}_A | \mathcal{F}_t\right].$$

Unlike the martingales considered earlier, orthogonality of X^A and D ensures that there is no change of probability for the event risk or that

$$Q(A) = P(A).$$

Orthogonality does ensure no risk premium on the event risk security itself. This is a consequence of the following equalities.

$$\begin{aligned} E^Q\left[\mathbf{1}_A\right] &= E^P\left[D(T)\mathbf{1}_A\right] \\ &= E^P\left[X^A(T)D(T)\right] \\ &= E^P[X^A(0)D(0) + \langle X^A, D\rangle_T] \end{aligned}$$

Table 1 Structure of event risk examples

	Two dimensions	One dimension
$D(t) = \mathcal{E}(L)_t$	$L(t)$ $= \int_0^t U(s)\mathrm{d}U(s) + V(s)\mathrm{d}V(s)$ $= \frac{1}{2}\left(R^2(t) - 2t\right)$	$L(t)$ $= \lambda \int_0^t \mathbf{1}_{B(s)>0}\mathrm{d}B(s)$ $= \lambda\left(B(t)^+ - \frac{1}{2}\Sigma(t)\right)$
$A = (\widehat{\gamma}_T > 0)$ with $\widehat{\gamma}$ a Brownian motion defined via	$\Theta(t)$ $\underset{\text{Def}}{=} \int_0^t -U(s)\mathrm{d}V(s) + V(s)\mathrm{d}U(s)$ $= \widehat{\gamma}\left(\int_0^t R(s)^2\mathrm{d}s\right)$	$\Theta(t)$ $\underset{\text{Def}}{=} B(t)^- - \frac{1}{2}\Sigma(t)$ $= \widehat{\gamma}\left(\int_0^t \mathbf{1}_{B(s)<0}\mathrm{d}s\right)$

$$\begin{aligned}
&= E^P[X^A(0)] + E^P\left[\langle X^A, D\rangle_T\right]\\
&= E^P[X^A(T)] + E^P\left[\langle X^A, D\rangle_T\right]\\
&= E^P\left[\mathbf{1}_A\right] + E^P\left[\langle X^A, D\rangle_T\right].
\end{aligned}$$

However even in the presence of zero correlation continuously in that $\langle X^A, D\rangle_t \equiv 0$ we may have the absence of the NKP property for the martingale X^A. Financially we may be interested in the structure of excess returns on derivatives written on the price process

$$J^A(t) = E^Q\left[\mathbf{1}_A|\mathcal{F}_t\right]$$

We show here that such derivatives may experience a non zero excess return even under orthogonality and even when $J^A(t) = X^A(t)$ and there is no risk premium on the base security.

We give in this section two examples of this situation. The first is in a two dimensional Brownian setting while the second considers the one dimensional case. The design of these examples is comparable and we present their structure together. Table 1 below presents the choice of D and the set A for both contexts.

In both cases, we first show that $X^A(t)$ may be represented as a stochastic integral with respect to $\mathrm{d}\Theta$ which is orthogonal to L. For this purpose we begin to express $\mathbf{1}_{\widehat{\gamma}(T)>0}$ as a stochastic integral with respect to $d\widehat{\gamma}$. Specifically we write

$$\mathbf{1}_{\widehat{\gamma}(T)>0} = \frac{1}{2} + \int_0^T \phi(s, \widehat{\gamma}(s))\mathrm{d}\widehat{\gamma}(s),$$

where

$$\phi(s,x) = \frac{1}{\sqrt{T-s}}\, n\left(\frac{x}{\sqrt{T-s}}\right)\mathbf{1}_{s<T}.$$

The computation of ϕ is comparable to the one of $q(s,x)$ in Example 2. Making the change of variable $s = H(u)$ where in the 2 dimensional case

$$H(u) = \int_0^u R^2(v)\mathrm{d}v$$

and in the one dimensional case

$$H(u) = \int_0^u \mathbf{1}_{B(v)<0}\mathrm{d}v$$

we obtain

$$\mathbf{1}_{\widehat{\gamma}(T)>0} = \frac{1}{2} + \int_0^\infty \phi(H(u), \widehat{\gamma}(H(u)))\mathbf{1}_{H(u)<T}\mathrm{d}\widehat{\gamma}(H(u))$$

which provides us with a formula for $(X^A(t), t \geq 0)$ in both cases:

$$X^A(t) = \frac{1}{2} + \int_0^t \phi(H(u), \Theta(u))\mathbf{1}_{H(u)<T}\mathrm{d}\Theta(u)$$

In order to show that the martingale X^A fails the NKP property it remains to show that the laws of $\left(\phi^2(H(u), \Theta(u))\mathbf{1}_{H(u)<T}\frac{\mathrm{d}}{\mathrm{d}u}\left(\langle\Theta\rangle(u)\right), u \geq 0\right)$ under Q and under P differ. Some elementary manipulations reduce the problem to showing that the laws of $(H(u), u \geq 0)$ differ under P and Q and this is easily verified and was referred to in previous examples.

7 Conclusion

This paper documents the problems inherent in viewing risk pricing in purely covariation terms. We provide examples of processes correlated with the change of measure density for which there is no change of probability on the reduced self sufficient filtration containing the risk being considered. Hence there is no excess return on derivative claims written on these processes provided the self sufficient filtration under P remains self sufficient under Q. This property is called almost no kernel pricing, ANKP as it falls just short of NKP which also requires the invariance of self sufficiency to the change of measure. However, we show that the loss of self sufficiency under Q when we have it under P requires a very specially designed kernel with exposure to the martingales of the self sufficient filtration that vanish on projection onto this filtration and are therefore a form of hidden exposures. We also provide examples with zero correlation where the risks involved are priced and we have excess returns associated with the investments involved.

These examples lead us to investigate the precise form of no kernel pricing in terms of correlation considerations. There are two sets of correlation conditions involved. The first delivers ANKP and introduces the concept of the self sufficient filtration containing a particular risk filtration. The self sufficient filtration heuristically includes all variables useful in predicting the future path of the risk in question. Hence we have the smaller filtration which we call the risk filtration that is contained in the larger self sufficient filtration containing the risk filtration that plays a critical role in understanding the pricing of risks. For ANKP to hold, the projection of the pricing kernel onto the self sufficient filtration must be orthogonal to all the martingales in the self sufficient filtration that end in a random variable measurable with respect to the smaller risk filtration.

The second set of correlation conditions are designed to ensure that self sufficiency is maintained under the measure change, a property critical for ANKP to actually yield NKP. For NKP it is required that the stochastic logarithm of the density minus the stochastic logarithm of the projection be orthogonal to all the martingales in the self sufficient filtration. Equivalently, the projection of the difference must vanish and hence we call these exposures hidden exposures.

Additionally we show that the density is strongly negatively priced in that, for example, all monotonic increasing functions of the density receive a negative risk premium. Furthermore, the equivalence of the density under increasing monotonic transformations is the unique random variable with this property. It is therefore the classic insurance asset.

A number of interesting questions remain. We would like to know if there exists a change of measure in a continuous time setting with the property that all nontrivial martingales on the filtration of the economy experience a change of probability or that all risks are priced. For the case of a one factor model with a constant price for the Brownian motion risk (Example 5) we showed that there exists a semimartingale that undergoes no change of probability. It remains to find a martingale if any, with this property.

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